This week

1. Section 12.4: the cross product
2. Section 12.5: lines and planes in space
The cross product – introduction

**Definition**

Let \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) be two vectors in \( \mathbb{R}^3 \). The cross product \( u \) and \( v \) is defined as

\[
\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2, \ u_3 v_1 - u_1 v_3, \ u_1 v_2 - u_2 v_1) .
\]

- The Dutch name for the cross product is *uitproduct* or *uitwendig product*.
- The cross product can be computed using this trick:

\[
\mathbf{u} \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} \times \mathbf{v} = \begin{pmatrix} u_1 \times v_2 & u_2 \times v_3 & u_3 \times v_1 \\ u_1 \times v_3 & u_2 \times v_1 & u_3 \times v_2 \end{pmatrix}
\]

**Laws and properties**

**Theorem**

For all \( u, v, w \in \mathbb{R}^n \) and \( r, s \in \mathbb{R} \) we have

1. \( (ru) \times (sv) = (rs)(u \times v) \)
2. \( u \times (v + w) = u \times v + u \times w \)
3. \( u \times v = -(v \times u) \)
4. \( (v + w) \times u = v \times u + w \times u \)
5. \( 0 \times u = u \times 0 = 0 \)
6. \( u \times (v \times w) = (u \cdot w)v - (u \cdot v)w \)

- Property 4 can be proved with properties 2 and 3.
Theorem

Let \( \mathbf{u} \) and \( \mathbf{v} \) be two vectors. If \( \theta \) is the acute positive angle between \( \mathbf{u} \) and \( \mathbf{v} \), then

\[
|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta.
\]

- Acute means: \( \theta \leq \pi \), hence \( \sin \theta \geq 0 \).

The cross product – geometry

Theorem

For all vectors \( \mathbf{u} \) and \( \mathbf{v} \) we have \( \mathbf{u} \times \mathbf{v} \perp \mathbf{u} \) and \( \mathbf{u} \times \mathbf{v} \perp \mathbf{v} \).

- Vector \( \mathbf{u} \times \mathbf{v} \) is perpendicular to the plane through \( \mathbf{u} \) and \( \mathbf{v} \).
- The length of \( \mathbf{u} \times \mathbf{v} \) is \( |\mathbf{u}| |\mathbf{v}| \sin \theta \).
- The right-hand rule determines the direction of \( \mathbf{u} \times \mathbf{v} \).
The area of a parallelogram

**Theorem**

Let \( u \in \mathbb{R}^3 \) and \( v \in \mathbb{R}^3 \) be the edges of a parallelogram \( P \). Then the area of \( P \) is equal to \( |u \times v| \).

![Parallelogram diagram]

- Observe that \( \sin \theta = \frac{h}{|v|} \), so \( h = |v| \sin \theta \).
- The area of \( P \) is
  \[
  |u| h = |u| |v| \sin \theta = |u \times v|.
  \]

**Example**

*Find the area of the triangle \( D \) with vertices \( P = (1,-1,0) \), \( Q = (2,1,-1) \) and \( R = (-1,1,2) \).*

- The triangle is one half of a parallelogram with edges \( \overrightarrow{PQ} \) and \( \overrightarrow{PR} \), hence the area of \( D \) is
  \[
  \frac{1}{2} \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right|.
  \]
- For the cross product we have
  \[
  \overrightarrow{PQ} \times \overrightarrow{PR} = (1,2,-1) \times (-2,2,2) = (6,0,6).
  \]
- For the area we have
  \[
  \text{area}(D) = \frac{1}{2} \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right| = \frac{1}{2} \sqrt{36 + 36} = 3\sqrt{2}.
  \]
The area of a parallelogram in $\mathbb{R}^2$

**Theorem**

Let $P$ be the parallelogram spanned by $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. Then $\text{area}(P) = |u_1v_2 - u_2v_1|$.

By appending a zero to the vectors $\mathbf{u}$ and $\mathbf{v}$ we can embed $P$ in $\mathbb{R}^3$:

- $\mathbf{u}' = (u_1, u_2, 0)$ and $\mathbf{v}' = (v_1, v_2, 0)$
- The area of $P$ is $\text{area } P = |\mathbf{u}' \times \mathbf{v}'| = |(0, 0, u_1v_2 - u_2v_1)| = |u_1v_2 - u_2v_1|$.

Distance to a line

**Problem**

Let $S$ be a point in space and let $\ell$ be a line through $P$ with direction vector $\mathbf{v}$. Find the distance $d$ of $S$ to $\ell$.

**Method 1:** Use the projection of $\mathbf{u} = \overrightarrow{PS}$ on $\ell$:

Works in $\mathbb{R}^n$ for every $n$

$$d = |\mathbf{h}| = \left| \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} \right|$$

**Method 2:** Use the cross product:

Only works in $\mathbb{R}^3$

$$d = |\mathbf{u}| \sin \theta = \frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{v}|}.$$
Example

*Find the distance of* \( S = (1, 1, 5) \) *to the line*

\[ \ell : \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t. \]

**Using method 2:**

- Define \( P = (1, 3, 0) \), \( \overrightarrow{OP} = (1, 3, 0) \) and \( \mathbf{v} = (1, -1, 2) \), then \( \ell : \mathbf{p} + t \mathbf{v} \ (t \in \mathbb{R}) \).
- Define \( \mathbf{u} = \overrightarrow{PS} = (1, 1, 5) - (1, 3, 0) = (0, -2, 5) \).
- \( \mathbf{v} \cdot \mathbf{v} = 1^2 + (-1)^2 + 2^2 = 6 \), hence \( |\mathbf{v}| = \sqrt{6} \).
- \( \mathbf{u} \times \mathbf{v} = (0, -2, 5) \times (1, -1, 2) = (1, 5, 2) \).
- The distance is
  \[
  d = \frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1^2 + 5^2 + 2^2}}{\sqrt{6}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.
  \]
**Definition**

A **parametrisation** of the plane $M$ is a function of the form

$$p + sv + tw, \quad s, t \in \mathbb{R}$$

- The vector $p$ is called a **support vector** and the vectors $v$ and $w$ are called **direction vectors**.

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**Example**

Find a parametrisation of the plane through the points $A = (0, 0, 1)$, $B = (2, 0, 0)$ and $C = (0, 3, 0)$.

- Choose support vector $a = \overrightarrow{OA} = (0, 0, 1)$.
- Choose direction vectors
  $$v = \overrightarrow{AB} = (2, 0, -1)$$
  and
  $$w = \overrightarrow{AC} = (0, 3, -1)$$
- A parametrisation then is
  $$r(s, t) = a + sv + tw$$
  $$= (0, 0, 1) + s(2, 0, -1) + t(0, 3, -1)$$
  $$= (2s, 3t, 1 - s - t), \quad s, t \in \mathbb{R}.$$
- Check: $A = r(0, 0)$, $B = r(1, 0)$ en $C = r(0, 1)$. 
**Problem**

Find an equation of a plane $M$ given by a parametrisation

$$p + sv + tw,$$

where $P$ is a point of $M$ and $p = \overrightarrow{OP}$.

**Method 1**: Three-point method: observe that $P$, $Q = p + v$ and $R = p + w$ are three points of $M$. This gives three equations involving $x$, $y$, $z$, $s$ and $t$. Eliminate $s$ and $t$ to find one equation in $x$, $y$ and $z$.

**Method 2**: Compute a normal vector $n = v \times w$ of $M$, then

$$M: n \cdot (x - p) = 0.$$ 

**Example**

Find an equation of the plane through the points $A = (0, 0, 1)$, $B = (2, 0, 0)$ and $C = (0, 3, 0)$.

- A parametrisation of $M$ is
  
  $$p + sv + tw = (0, 0, 1) + s(2, 0, -1) + t(0, 3, -1)$$

- Find a normal vector:
  
  $$(2, 0, -1), 2, 0$$

  $$n = v \times w = \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = (3, 2, 6).$$

- The normal equation of $M$ is
  
  $$n \cdot (x - p) = 0$$

  $$(3, 2, 6) \cdot ((x, y, z) - (0, 0, 1)) = 0$$

  $$3x + 2y + 6(z - 1) = 0$$

  $$3x + 2y + 6z = 6$$
**Theorem**

*Two different non-parallel planes intersect in a line.*

- Non-parallel means: the normals of both planes have different directions.
- If the planes are called $M$ and $N$, then the intersection line is denoted as follows:
  \[ \ell = M \cap N. \]
- A line in space can be regarded as the intersection line of two planes, in other words: it is the solution of a system of two equations:
  \[
  \ell: \begin{cases}
  ax + by + cz = d, \\
  px + qy + rz = s.
  \end{cases}
  \]

**Example**

*Find a parametrisation of the intersection line of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.*

**Method 1:**

- From the first equation follows $x = 2y + \frac{2}{3}z + 5$.
- Substitution in the second equation gives
  \[ 2\left(2y + \frac{2}{3}z + 5\right) + y - 2z = 5, \]
  and after simplification we have
  \[ z = \frac{15}{2}y + \frac{15}{2}. \]
- Choose one of the unknowns as parameter. For example, let $y = t$, then
  \[ z = \frac{15}{2}t + \frac{15}{2} \quad \text{and} \quad x = 2t + \frac{2}{3}\left(\frac{15}{2}t + \frac{15}{2}\right) + 5 = 7t + 10. \]
- A parametrisation of the intersection line is
  \[ \mathbf{r}(t) = \left(7t + 10, t, \frac{15}{2}t + \frac{15}{2}\right) = \left(10, 0, \frac{15}{2}\right) + t\left(7, 1, \frac{15}{2}\right), \quad t \in \mathbb{R}. \]
Method 2:

- The normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are perpendicular to the intersection line, so the cross product of $\mathbf{n}_1$ and $\mathbf{n}_2$ is a direction vector of the intersection line.
- Extract the normal vectors from the equations:
  - $M_1: 3x - 6y - 2z = 15$, $\rightarrow \mathbf{n}_1 = (3, -6, -2)$,
  - $M_2: 2x + y - 2z = 5$, $\rightarrow \mathbf{n}_2 = (2, 1, -2)$,

  hence $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = (14, 2, 15)$.

A support vector can be found by choosing a value for $x$, $y$ or $z$, and then solving both equations for $x$ and $y$. For example, choose $y = 0$:

- $3x - 2z = 15$,
- $2x - 2z = 5$.

Subtracting both equations gives $x = 10$, and therefore $z = \frac{15}{2}$.

A support vector is $\mathbf{p} = (10, 0, \frac{15}{2})$.

A parametrisation of the intersection line is

$$\mathbf{p} + t\mathbf{v} = \left(10, 0, \frac{15}{2}\right) + t \left(14, 2, 15\right)$$

$$= \left(10, 0, \frac{15}{2}\right) + 2t \left(7, 1, \frac{15}{2}\right).$$
Assignment: IMM2 - Tutorial 8.2